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A CRACK AT THE INTERFACE OF ANISOTROPIC BODIES. SINGULARITIES OF THE ELASTIC FIELDS AND A CRITERION FOR FRACTURE WHEN THE CRACK SURFACES ARE IN CONTACT†

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Novozhilov's quasistatic brittle fracture criterion is used to determine the direction angle of the branch of a crack with closed surfaces at the interface of two anisotropic bodies. The asymptotic form of the elastic fields near the crack tip is investigated and corollaries of Novozhilov's criterion are derived in an asymptotic formulation. Normalizations of the singular solutions, which have been adapted to the force and deformation fracture criteria, are discussed. In particular, it is established that, in the case of general anisotropy, there is no oscillation of the crack surfaces which can appear in the solution of the linear problem for an open crack.

1. FORMULATION OF THE PROBLEM

Consider a planar composite body $\Omega = \Omega^+ \cup \Omega^-$, each of the parts of which is homogeneous and anisotropic. Along the interface line $\Upsilon^0 = \partial\Omega^+ \cap \partial\Omega^-$, which lies on the x_1 axis of Cartesian coordinates $x = (x_1, x_2)$, there is a crack (a cut) M^0 which, to be specific, is a boundary crack. We will place the origin of coordinates O at the crack tip and denote the polar coordinates with polar axis directed along the prolongation of the crack by (r, φ), that is, $r = |x|$ and $\varphi \in (-\pi, \pi)$. We will assume that there is complete bonding of the materials in the section $\Gamma^0 = \Upsilon^0 \backslash \overline{M^0}$. The displacement vector $u^\pm = (u_1^\pm, u_2^\pm)$ and the normal stress vector $\sigma^{(2)\pm} = (\sigma_{21}^\pm, \sigma_{22}^\pm)$ are continuous.

It is well known (see [1–9], etc) that the power solutions (the displacement fields)

$$U(x) = r^\Lambda \Phi(\varphi) \tag{1.1}$$

of the model problem of a composite plane with a semi-infinite cut, which describe the behaviour of the elastic fields close to the tip of an open crack, can have complex exponents $\Lambda = \pm i\gamma + 1/2$, $\gamma > 0$. In this case, the solution of the linear problem in the theory of elasticity is characterized by the overlapping of the crack surfaces for any sign of the stress intensity factor. This makes its physical interpretation difficult, and is indicative of the need for a complete (non-linear) solution of the Signorini problem which includes the formulation of unilateral bonds and enables us to determine the contact set which is previously unknown. However, there are also other approaches to eliminating the above-mentioned contradiction. The first approach [10, 11] ignores the overlapping of the crack surfaces since, according to calculations, in many cases it is localized in an extremely small neighbourhood of the tip, far from which the deviation from the contact solution is insignificant. The second approach [12–14] also retains the linear defining relations but artificially introduces a contact zone at the mouth of the crack and suggests a technique for calculating the length of this zone for certain specific problems concerning a composite isotropic plane with a cut. In this paper, an analogous situation is considered

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and it is simply assumed that, at the mouth, the crack surfaces overlap. However, the subject of the investigation is the singularities in the elastic fields and their effect on the quasistatic fracture process.

In order to answer many of the attendant questions, it suffices to study the model problem of a composite plane with a semi-infinite cut $M = \{x: x_1 \le 0, x_2 = 0\}$, that is

$$-\partial_1\sigma^{\pm}_{1k}(u^{\pm}; x) - \partial_2\sigma^{\pm}_{2k}(u^{\pm}; x) = 0, \quad x \in \mathbb{R}^2_{\pm} = \{x: \pm x_2 > 0\} \tag{1.2}$$

$$u^+_k(x_1, 0) = u^-_k(x_1, 0), \;\; \sigma^+_{2k}(u^+; x_1, 0) = \sigma^-_{2k}(u^-; x_1, 0), \;\; x_1 \in \mathbb{R}^1_+ = (0, +\infty); \;\; k = 1, 2 \tag{1.3}$$

$$\begin{gathered} u^+_2(x_1, 0) = u^-_2(x_1, 0), \;\; \sigma^+_{22}(u^+; x_1, 0) = \sigma^-_{22}(u^-; x_1, 0), \;\; \sigma^{\pm}_{12}(u^{\pm}; x_1, 0) = 0, \\ x_1 \in \mathbb{R}^1_- = (-\infty, 0) \end{gathered} \tag{1.4}$$

Here

$$\sigma^{\pm}(u^{\pm}) = A^{\pm}\varepsilon(u^{\pm}) \tag{1.5}$$

$\sigma^{\pm}_{jk}$ are the Cartesian components of the stress tensor, $\sigma^{\pm}$ and $A^{\pm}$ are the elastic moduli tensors, which are constant and possess the usual properties of positiveness and symmetry, and $\varepsilon(u)$ is the strain tensor with the components $\varepsilon_{jk}(u) = (\partial_j u_k + \partial_k u_j)/2$ or $\partial_j = \partial/\partial x_j$ ($j = 1, 2$). If $A^+ = A^-$ in Hooke's law (1.5), then the plane is found to be homogeneous, and the total bonding conditions (1.3) can be omitted. Relations (1.4) mean that the surfaces $M_{\pm}$ of a semi-infinite crack are in contact without friction. We emphasize that the following condition of "compressive stresses", which attributes a physical meaning to the solution of problem (1.2)–(1.4), is required in the *a posteriori* verification

$$\sigma^+_{22}(u^+; x_1, 0) = \sigma^-_{22}(u^-; x_1, 0) \le 0, \quad x_1 \in \mathbb{R}^1_- \tag{1.6}$$

In Section 2 the exponents of the possible power solutions are determined (the integers and half-integers are real numbers, that is, no oscillation of the closed crack surfaces occurs during loading, but this does not contradict the formulation of the problem) and the number of linearly independent solutions (1.1) with the same exponents is found. Normalizations of the singular solutions are discussed in Section 3 and it is verified that the main singularity in the stresses in a homogeneous plane satisfies the physical condition (1.6) (the left-hand side is equal to zero) and a hypothesis is proposed concerning the conditions under which it holds in the case of the composite plane. Sections 4 and 5 are concerned with the application of the fracture criterion proposed by Novozhilov in [15].

We will now explain the reasons for choosing this criterion. The introduction of a time-like loading parameter, which is necessary from the very beginning when formulating the *quasistatic* fracture problem, requires an *a posteriori* analysis of the shape of the free surface which has been formed anew. Hence, in view of the contact of the crack surfaces which is postulated, the model problem of a crack with a small germ has to be solved in a complete formulation which admits of the surfaces of the main crack to be separated and the surfaces of the Branch crack to come in contact. An explicit solution or detailed information about it is required, and these are unavailable up to the present time. The methods [16] of analysing Signorini problems for bodies with cracks and the formulae in [17, 18] for the increment in the potential energy of deformation as a consequence of the propagation of a rectilinear crack with contacting surfaces do not enable one to determine the direction angle of the crack branch. Moreover, it follows from the subsequent results that, in the case of an isotropic body, a shear load, which generates singularities in the stresses at the tip, leads to a deflection of the crack but, under normal loading, which stimulates rectilinear development of a crack, the stresses remain bounded and the leading term in the energy increment [17, 18] becomes zero.

Novozhilov's criterion remains an *a priori* criterion even after the introduction of a time-like parameter, and this property of it is used to determine the required angle and to predict the possibility of branching. Furthermore, the well-known criteria for the maximum breaking stresses and the absence of shear stresses follow from the asymptotic formulation [19] of the criterion (see Section 5).

2. POWER SOLUTIONS

In the case of arbitrary anisotropy, a direct calculation of the exponents Λ in formulae (1.1) is hardly possible. An approach [20] (see also [21, Ch. 7], and [22, Section §7]) has been suggested which, for an extensive class of self-adjoint elliptic boundary-value problems, enables one to determine the exponents Λ for all possible power solutions (1.1) (but not their angular parts Φ). Initially developed for differential operators with constant coefficients, it was subsequently adapted [9] to the case of piecewise-constant coefficients with discontinuities in the prolongation of the crack. To sum up, it was established that the exponents Λ are either integers $\Lambda = m$ or they have the form $\Lambda = i\gamma_n + m/2$ and, moreover, the set of imaginary parts $\{\gamma_1, \dots, \gamma_N\}$ is independent of $m \in \mathbb{Z} = \{0, \pm1, \pm2, \dots\}$. The shortcoming of the approach discussed in [20] lies in the impossibility of determining the conditions under which all the exponents Λ turn out to be real (what has been said does not, however, refer to problem (1.2)–(1.4)). The fact that the exponents Λ are real for a crack in a homogeneous anisotropic solid was proved for the first time in [23] using an analysis of the corresponding boundary integral equations. Results which are similar in their formulation but more profound in the case of general elliptic systems with the same boundary conditions on the crack surfaces were established later in [24]. At the same time, the methods developed in [24] are not suitable for a composite solid.

The method we proposed in [9, 20] is based on certain considerations, the main one of which can be referred to as *differentiation along a crack*: the derivative of power solution (1.1)

$$\partial_1 U(x) = r^{\Lambda - 1}\Psi(\varphi) \tag{2.1}$$

remains a power solution of problem (1.2)–(1.4), and $\Psi \neq 0$ provided U is not a function of the single variable x_2. Together with the general results of the theory of boundary-value problems in domains with piecewise-smooth boundaries (see the key papers [25, 26] and, for example, the monograph [21]), linkage of the power solutions (1.1) and (2.1) leads to the following relation for the set Σ of exponents of the non-trivial power solutions

$$\Lambda \in \Sigma \Rightarrow -\bar{\Lambda} \in \Sigma, \quad \Lambda \in \Sigma \Rightarrow \bar{\Lambda} \in \Sigma, \quad \Lambda \in \Sigma \Rightarrow \Lambda - 1 \in \Sigma \tag{2.2}$$

(the bar denotes a complex conjugate). We note that the first two expressions of (2.2) follow respectively from the formal self-adjointness of problem (1.2)–(1.4) and from the fact that the coefficients are real (the tensors $A^{\pm}$ in Hooke's law (1.5)). The last assertion of (2.2) is required in the additional line of reasoning, which will be presented below.

A further important aspect is the polynomial property [22] of a system in the theory of elasticity. The elastic energy functional only degenerates for rigid displacements which are vector polynomials of the variables x_1 and x_2. This guarantees the following assertion (see [21, Theorem 6.1.2] and [22, Assumption (2.3)])

$$\Lambda \in \Sigma, \quad \mathrm{Re}\Lambda = 0 \Rightarrow \Lambda = 0 \tag{2.3}$$

At the same time, the power solutions (1.1) with a zero exponent Λ are rigid translational displacements, in particular, the unit vectors $e^1 = (1, 0)$ and $e^2 = (0, 1)$. Furthermore, there are exactly two linearly independent logarithmic-power solutions

$$e^j \ln r + \mathcal{T}^j(\varphi), \quad j = 1, 2 \tag{2.4}$$

They generate forces concentrated at the point O. Taking account of formulae (2.2) and (2.3), we will now show that only the real exponents, and this means the integer exponents of the non-trivial power solutions (1.1), are located on the lines $\{\Lambda \in \mathbb{C} : \mathrm{Re}\Lambda = m \in \mathbb{Z}\}$.

The next stage is to verify that none of the exponents being discussed are in the strip $\{\Lambda \in \mathbb{C} : 0 < \mathrm{Re}\Lambda < 1/2\}$. In the case of problem (1.2)–(1.4), it literally repeats the arguments presented earlier [9, Section 4] and will not be reproduced here. We note that, here, use is made of assertions (2.2), the integral formulae for the stress intensity factor and, also, the monotonic decrease in the potential energy of deformation functional in the case of a growing crack, but for a fixed load. Returning once again to the technique of differentiation along a crack, we conclude that

$$\Sigma = \{m, m \pm i\gamma_n + 1/2 : m \in \mathbb{Z}, n = 1, \dots, N\} \tag{2.5}$$

Here $\{\gamma_1, \dots, \gamma_N\}$ is a set of numbers which depends on the elastic moduli of the materials. It will be shown later than $N = 1$ and $\gamma_1 = 0$.

We will now determine the number of linearly independent power solutions for a fixed real part of the exponent $\lambda \in \Sigma$. The points $\lambda \in \Sigma$ are the eigenvalues of a certain operator pencil in the arc (see

[25], and, also, [21, §3.5] and [9, Section 2]. By a continuous change in the tensors $(-\pi, \pi)$ in Hooke's law (1.5), this pencil is transformed into the pencil corresponding to problem (1.2)–(1.4) concerning an isotropic, homogeneous plane with a cut M, for which the following is known (see, for example, [13, 27]): first, in the case when $\Lambda = 1$, there are three linearly independent solutions, which are given by the formula

$$r^1\Phi(\varphi) = (a_1x_1 - a_0x_2, a_2x_2 - a_0x_1), \quad a_q \in \mathbb{R} \tag{2.6}$$

and, second, if $\mathrm{Re}\Lambda = 1/2$, then $\Lambda = 1/2$, and any power solution with this exponent is proportional to solution (1.1) in which $\Lambda = 1/2$ and

$$\begin{aligned} \Phi_r(\varphi) &= \frac{1}{4\sqrt{2\pi}\mu}\left(3\sin\frac{3\varphi}{2} - (5-8\nu)\sin\frac{\varphi}{2}\right) \\ \Phi_\varphi(\varphi) &= \frac{1}{4\sqrt{2\pi}\mu}\left(3\cos\frac{3\varphi}{2} - (7-8\nu)\cos\frac{\varphi}{2}\right) \end{aligned} \tag{2.7}$$

Here, $\lambda \geq 0$ and $\mu > 0$ are Lamé constants, $\nu = \lambda[2(\lambda + \mu)]^{-1}$ is Poisson's ratio (we are dealing with plane deformation). Note that formulae (2.7) indicate the polar components of the angular part Φ and the vector (2.6) is written in a Cartesian system of coordinates. Finally, logarithmic-power solutions with exponents $\Lambda = 1$ and $\Lambda = 1/2$ do not exist.

Since, according to what has been proved previously, the eigenvalues Λ cannot leave the straight lines: $\{\Lambda: \mathrm{Re}\Lambda = m + 1/2\}$ or the set $\mathbb{Z}$, and the theorem in [28] concerning the preservation of the full multiplicity of the spectrum under a continuous change in the pencil asserts that, for arbitrary $A^\pm$ in the law (1.5), only a single exponent $\Lambda_{1/2}$ lies on the line $\{\Lambda: \mathrm{Re}\Lambda = 1/2\}$ and the exponent $\Lambda_1 = 1$ corresponds to three linearly independent power or logarithmic-power solutions. On account of the second assertion of (2.2), $\Lambda_{1/2} = 1/2$ (when $\Lambda_{1/2}$ is complex, a second exponent $\bar{\Lambda}_{1/2} \neq \Lambda_{1/2}$ will appear on the same line), and the third and first assertions of (2.2) establish that $\gamma_1 = \ldots = \gamma_N = 0$ and that only one power solution, that is, $N = 1$, corresponds to each half-integral exponent $\Lambda = m + 1/2$.

It remains to consider the integral points from the set (2.5). In the isotropic case, among the solutions (2.6), there is a rotation ($a_1 = a_2 = 0$) and, also, two solutions U_j for which $\sigma_{12}^\pm(U^j) = 0$ and $\sigma_{kk}^\pm(U^j) = \delta_{jk}$ $(j, k = 1, 2)$. Similar solutions can also be constructed in the case of an anisotropic composite plane. They are found to be piecewise linear and form a basis in the linear manifold of the power solutions with exponents $\Lambda = 1$. There are as many of them as in the isotropic situation, that is, other solutions, in particular, logarithmic-power solutions with the exponent $\Lambda = 1$, do not exist according to what has been proved above. Consequently, there are also no logarithmic-power solutions in the case of the exponents $\Lambda = m > 1$ (the $\ln r$ factor cannot be eliminated by the differentiation $\partial^{m-1}/\partial x_1^{m-1}$). Among the named piecewise-linear solutions, there is one which depends solely on x_2 and which vanishes after differentiation with respect to x_1, for example, $a_0 = a_1 = 0$ in the definition of (2.6). Only two, and not three, polynomial solutions e^1 and e^2 can therefore be found for the exponent $\Lambda = 0$. Solutions (2.4), which contain $\ln r$, are not obtainable from the other solutions by differentiation along the crack.

Suppose $m \neq 0, 1$, $a^\pm \in \mathbb{R}^2$ and $U^\pm(x) = a^\pm x_2^m$ are solutions of problem (1.2)–(1.4). Then, according to the equilibrium equation (1.2), $\partial_2\sigma_{2k}^\pm(U^\pm) = 0$, and this means that the equalities $\sigma_{2k}^\pm(U^\pm) = 0$ $(k = 1, 2)$ hold which, together with the obvious formulae

$$\varepsilon_{11}(U^\pm) = 0, \quad \varepsilon_{12}(U^\pm) = \varepsilon_{21}(U^\pm) = ma_1^\pm x_2^{m-1}/2, \quad \varepsilon_{22}(U^\pm) = ma_2^\pm x_2^{m-1}$$

contradict the fact that the tensors $A^\pm$ are positive definite. In particular, this fact, together with formula (2.4), completes the verification of the third statement of (2.2). Furthermore, not one non-trivial solution with an exponent $\Lambda = m > 1$ is eliminated by the differentiation $\partial/\partial x_1$ and, therefore, by virtue of the general results in [26] and the third assertion of (2.2) (also, see [21, Assumptions 1.2.6 and 3.5.4]), exactly the same number of solutions (1.1) is found for $\Lambda = m$ and $\Lambda = -m$ as in the case when $\Lambda = 1$.

The main properties of the spectrum (2.5) have been studied. The set Σ consists of integral and half-integral numbers when $N = 1$ and $\gamma_1 = 0$ in formula (2.5). When $m \in \mathbb{Z}$, three linearly independent power solutions correspond to the exponent $\Lambda = m \neq 0$ and there are no logarithmic-power solutions. If $\Lambda = 0$, then solutions (1.1) are constant vectors, but there are a further two solutions (2.4). In the case when $\Lambda = m + 1/2$ which is discussed later, there is only one power solution (1.1).

3. THE NORMALIZATION OF THE SINGULAR SOLUTION

The solution of problem (1.2)–(1.4)

$$U(x) = r^{1/2}\Phi(\varphi) \tag{3.1}$$

with an exponent $\Lambda = 1/2$ and angular parts (2.7) corresponds to a second (shear) mode for a homogeneous isotropic plane with a crack. In this case,

$$\sigma_{22}^{\pm}(U^{\pm}; x_1, \pm 0) = 0, \quad x_1 < 0 \tag{3.2}$$

and the factor K in the asymptotic form of the elastic fields around the crack tip

$$u(x) = c + KU(x) + O(r), \quad \sigma(u; x) = K\sigma(U; x) + O(1), \quad r \to +0 \tag{3.3}$$

is found according to the classical definition of the stress intensity factor

$$K = K_{\mathrm{II}} = \lim_{r \to +0} (2\pi r)^{1/2}\sigma_{12}(u; r, 0) \tag{3.4}$$

The correctness of equality (3.2) is next verified in the case of an anisotropic homogeneous plane, and this means, by virtue of representations (3.3), that the normal stresses along the contact line remain bounded. In other words, for any load, the crack can be closed by the imposition of an additional sufficiently large uniform compressive stress field. Hence, in homogeneous mountain masses, where compressive stresses predominate, fracture occurs when the surfaces of cracks are in complete contact.

In order to verify the consistency of definition (3.4) in the case of a composite anisotropic plane, we will use the line of reasoning previously described in [9] and ascertain that it is possible to normalize the angular part Φ of the power solution (3.1) with the relation

$$\sigma_{12}^{\pm}(U^{\pm}; r, 0) = (2\pi r)^{-1/2} \tag{3.5}$$

If it is assumed that $\sigma_{12}^{\pm}(U^{\pm}; r, 0) = 0$, then the pair U^+, U^- is found to be a solution of equilibrium equation (1.2) with uniform mixed coupling conditions along the interface line

$$U_2^+(x_1, 0) = U_2^-(x_1, 0), \ \sigma_{22}^+(U^+; x_1, 0) = \sigma_{22}^-(U^-; x_1, 0), \ \sigma_{12}^{\pm}(U^{\pm}; x_1, 0) = 0, \ x_1 \in \mathbb{R} \tag{3.6}$$

In view of the polynomial property [22] and the formal self-adjointness, problem (1.2), (3.6) is an elliptic problem, that is, the solution $U = U^{\pm}$, which is bounded in the neighbourhood of the point O, is found [29] to be smooth, and therefore does not possess a singularity $O(r^{1/2})$. The resulting contradiction establishes that the shear stresses do not vanish along the crack, that is, condition (3.5) can always be complied with.

The normalization (3.5), which has been adapted to the *force* criteria of fracture, can be replaced by the following normalization which accompanies the *deformation* criterion of fracture

$$[U_1](-r) = 8(2\pi)^{-1/2}br^{1/2} \tag{3.7}$$

Here, $[U_k](x_1) = U_k^+(x_1, 0) - U_k^-(x_1, 0)$ is the discontinuity in the displacements on the crack surfaces M, and $b = (B_{11,11}^+ + B_{11,11}^-)/2$ and $B_{jk,pq}^{\pm}$ are elements of the compliance tensors, which are inverse to the rigidity tensors $A^{\pm}$ in Hooke's law (1.5). In the case of an isotropic homogeneous plane,

$$b = B_{11,11}^{\pm} = [4\mu(\lambda + \mu)]^{-1}(\lambda + 2\mu) \tag{3.8}$$

and the factor on the right-hand side of equality (3.7), which had originally been missing in [30], is chosen such that the angular parts (2.7) satisfy condition (3.7) with the coefficient (3.8).

We can now explain the reason for the behaviour of the element $G_{11,\,11}$ in formula (3.7) and the unexpected behaviour of $G_{12,\,12}$. Suppose the plane is homogeneous, that is, $A^+ = A^-$ in Hooke's law (1.5). By virtue of relations (3.1) and (3.7)

$$[\sigma_{11}(U)](-r) = \sigma_{11}^0 r^{-1/2}, \quad r^{-1/2}[\varepsilon_{11}(U)](-r) = -4(2\pi)^{1/2} G r^{-1/2}$$

We emphasize that $[\sigma_{2k}(U)](-r) = 0$ $(k = 1, 2)$ according to the coupling conditions (1.3). Denoting the second rank tensor with the Cartesian components $e_{11} = 1$ and $e_{jk} = 0$ when $j + k > 2$ by e, we calculate the factor σ_{11}^0

$$\sigma_{11}^0 r^{-1/2} e = [\sigma(u)](-r) = A[\varepsilon(U)](-r)$$

$$\sigma_{11}^0 G_{11,\,11} r^{-1/2} = \sigma_{11}^0 r^{-1/2} e A^{-1} e = e[\varepsilon(U)](-r) = [\varepsilon_{11}(U)](-r)$$

So, $\sigma_{11}^0 = -4(2\pi)^{-1/2}$ and the normalization (3.7) is equivalent to the following

$$[\sigma_{11}(U)](-r) = -4(2\pi r)^{-1/2} \tag{3.9}$$

Since a root singularity in the stresses is characterized by just a single stress intensity factor and the normalization (3.5) and (3.7) are in fact equivalent, the fracture criteria, operating with the stress intensity factor K can be considered as both force and deformation criteria. Moreover, according to equality (3.9) for a homogeneous solid, the stress intensity factor L, which corresponds to the "deformation normalization" (3.7), is defined as

$$L = -\frac{1}{4} \lim_{r \to +0} (2\pi r)^{1/2} [\sigma_{11}(u)](-r) \tag{3.10}$$

We emphasize that, on the right-hand side of (3.10), there is a discontinuity in the stresses $\sigma_{11}(u)$ on the crack surfaces which, according to what has been proved, does not vanish.

In the case of an open crack, that is, when the coupling conditions (1.3) are replaced by the boundary conditions

$$\sigma_{2k}^{\pm}(u^{\pm}; x_1, 0) = 0, \quad x_1 \in \mathbb{R}_+^1, \quad k = 1, 2 \tag{3.11}$$

there are [9] two solutions, U^1 and U^2, of the form of (1.1) with exponents $\Lambda = \pm i\gamma + 1/2$. If $\gamma = 0$ and there is no overlapping of the crack surfaces, then, using the reasoning which led to formula (3.5), we can convince ourselves of the possibility of the deformation normalization of the basis $\{U^1, U^2\}$ of power solutions (1.3) of problem (1.2), (1.4), (3.11):

$$[U_k^j](-r) = 8(2\pi)^{-1/2} b r^{1/2} \delta_{j,\,3-k} \tag{3.12}$$

By virtue of boundary conditions (3.11) and equalities (3.12) when $j = k = 2$, the solution U^2, which corresponds to the shear mode, satisfies relations (1.2)–(1.4) and (3.2), and this means that it also satisfies he physical requirement (1.6).

If, however, $\gamma \neq 0$, then not one non-trivial linear combination $W = c_1 U^1 + c_2 U^2$ can have a zero discontinuity $[\sigma_{22}(W)](-r)$ on the crack surfaces. Actually, the field W would otherwise be found to be a solution of problem (1.2)–(1.4), but possesses complex homogeneity exponents Λ which would contradict what has been proved in Section 2. On the other hand, when $\gamma \neq 0$, solution (3.1) of problem (1.2)–(1.4) does not satisfy condition (3.2), or else it becomes a solution of problem (1.2), (1.4), (3.11) with an exponent $\Lambda \neq \pm i\gamma + 1/2$. The violation of equality (3.2) means that solution (3.3) only satisfied the physically correct condition of compressive stresses (1.6) in the case of a *specific sign* of the stress intensity factor K.

The transformations, carried out in the comment presented above, enable us to propose the following hypothesis: in the case of boundary conditions (3.11), all of the exponents Λ of the non-trivial power solutions (1.1) are real if and only if the equality

$$B_{11,\,11}^{+} = B_{11,\,11}^{-} \tag{3.13}$$

is satisfied. In the case of isotropic materials, condition (3.13) is identical to Dundurs' condition [31].

4. NOVOZHILOV'S CRITERION

The criterion of quasistatic fracture

$$\frac{1}{d}\int_0^d \sigma_{\varphi\varphi}(r,\varphi)|_{\varphi=\theta}dr = \sigma_c \tag{4.1}$$

proposed by Novozhilov [15] for determining the equilibrium state of cracks, was adapted in [19, 32–34] for finding critical loads in the case of different load concentrators. On the left-hand side of relation (4.1), integration is carried out over the segment $I(\theta) = \{x: r \in [0, d], \varphi = \theta\}$ of length d, starting from the tip O.

Since the material is not assumed to be isotropic and homogeneous, we assume that its characteristics at the point O $d = d(\theta)$ and $\sigma_c = \sigma_c(\theta)$, the characteristics size of the medium (the grain size [19], for example) and the critical stress (the theoretical strength [35]) depend on the direction $\theta \in (-\pi, \pi)$ of the section $I = I(\theta)$. Within each of the half-planes $\mathbb{R}^2_\pm$, that is, when $\pm\theta \in (0, \pi)$, it is reasonable to take these relations as being smooth, but discontinuities of the first kind are permitted in them at the point $\theta = 0$ (the interface line). If the bonding is unreliable and the theoretical strength of the materials in $\mathbb{R}^2_\pm$ significantly exceeds the magnitude of $\sigma_c(0)$, then rectilinear propagation of the crack is found to be preferable [36]. In the remaining cases, the question of the determination of the angle of deviation of the crack branch from the Ox_1 axis becomes paramount.

We will now consider a bounded composite body $\Omega = \Omega_+ \cup \Omega_-$ with a boundary crack M^0. Suppose that, when there are not bulk forces, a load $p(x; \tau)$ is applied to the external surface $\partial\Omega\backslash\Upsilon^0$, which depends on the dimensionless time-like parameter τ (which is strictly monotonic with respect to the real time t), and the rate of change of τ is assumed to be small compared with the propagation velocity of elastic waves divided by the characteristic dimension l of the body Ω (by the length of the crack, for example: not to be confused with the parameter $d \ll l$). This formulation enables us to neglect the inertia forces justifiably and to formulate the quasistatic fracture problem in the following manner: it is required to determine the instant $\tau = \tau_*$ at which equality (4.1) is satisfied for any angle θ but, when $\tau < \tau_*$, the left-hand side of (4.1) is strictly less than $\sigma_c(\theta)$ for any θ. The corresponding load $p(x; \tau_*)$ will be the critical load. We note that, in the case of simple loading, the above-mentioned formulation, which is suitable for many fracture criteria, can be derived from the dynamic fracture criterion [34]. In other words, the function

$$(-\pi, \pi) \ni \theta \mapsto F(\tau_*; \theta) - \sigma_c(\theta) \tag{4.2}$$

where

$$F(\tau; \theta) = \frac{1}{d(\theta)}\int_0^{d(\theta)} \sigma_{\varphi\varphi}(\tau; x)|_{\varphi=\theta}ds \tag{4.3}$$

must reach a global maximum (equal to zero) at one or several points but remains negative for all θ for $\tau < \tau_*$. In the case of a multiplicity of the zeroes of function (4.2), the appearance of several branches would be expected and one speaks of the branching of the crack.

We will now consider the case of a homogeneous ($A^+ = A^-$) plane for which the quantities $d(\theta)$ and $\sigma_c(\theta)$ depend continually on the angle θ. We fix θ and introduce Cartesian coordinates (s, n), directing the s axis along the section $I(\theta)$. Using equilibrium equations (1.2), rewritten in the new coordinates,

$$-\partial_s\sigma_{ss} - \partial_n\sigma_{ns} = 0, \quad -\partial_s\sigma_{sn} - \partial_n\sigma_{nn} = 0 \tag{4.4}$$

we transform the derivative F' of function (4.3) with respect to the variable θ in the following way

$$F'(\tau;\theta) = -\frac{d'(\theta)}{d(\theta)}F(\tau;\theta) + d'(\theta)\sigma_{nn}(\tau; d(\theta), \theta) + \frac{1}{d(\theta)}J(\tau;\theta)$$

$$J = \int_0^d \frac{\partial}{\partial\theta}\sigma_{nn}|_{n=0}ds = \int_0^d s\frac{\partial}{\partial n}\sigma_{nn}|_{n=0}ds = -\int_0^d s\frac{\partial}{\partial s}\sigma_{sn}|_{n=0}ds = \int_0^d \sigma_{sn}|_{n=0}ds - d\sigma_{sn}|_{s=d,\, n=0}$$

As a result, we obtain the relation

$$F'(\tau;\theta)=\frac{1}{d(\theta)}\int_0^{d(\theta)}\{\sigma_{ns}(\tau;x)-d'(\theta)\sigma_{ns}(\tau;x)\}|_{\varphi=0}ds-$$
$$-\{\sigma_{ns}(\tau;x)-d'(\theta)\sigma_{ns}(\tau;x)\}|_{r=d(\theta),\,\varphi=0}= \tag{4.5}$$
$$=\frac{1}{D(\theta)}\left\{\frac{1}{d(\theta)}\int_0^{d(\theta)}\sigma_{nN}(\tau;x)|_{\varphi=0}-\sigma_{nN}(\tau;x)|_{r=d(\theta),\,\varphi=0}\right\},\ D(\theta)=\left(1+\left[\frac{d'(\theta)}{d(\theta)}\right]^2\right)^{-1/2}$$

By $N = N(\theta)$, we mean a unit vector which is tangential to a graph of the function $r = d(\varphi)$ at the point $\varphi = \theta$. Its projections onto the s and n axes have the form

$$N_s = D(\theta),\quad N_n = -D(\theta)d'(\theta)/d(\theta)$$

So, in the formation of a branch, which starts out from the tip O in a direction θ, expression (4.5) at the instant $\tau = \tau_*$ is identical to the derivative $\sigma_c'(\theta)$, that is

$$\frac{1}{d(\theta)}\int_0^{d(\theta)}\sigma_{nN}(\tau_*;x)|_{\varphi=\theta}ds-\sigma_{nN}(\tau_*;x)|_{r=d(\theta),\,\varphi=0}=D(\theta)\sigma_c'(\theta) \tag{4.6}$$

If the strength properties are isotropic (the elastic properties of the body can still retain anisotropy), then $d'(\theta) = 0$, $\sigma_c'(\theta) = 0$, and the condition which has been found simplifies to the condition

$$\sigma_{r\varphi}(\tau_*;d,\theta)=\frac{1}{d}\int_0^d\sigma_{r\varphi}(\tau_*;r,\theta)ds \tag{4.7}$$

In other words, the shear stress at the end of the section $I(\theta)$ is identical to the mean value of this stress over the section. In the case of simple loading $p(x;\tau) = \tau p^0(x)$, the angle θ is independent of the loading instant, and the argument τ_* can be removed from relations (4.6) and (4.7).

Note that conditions (4.7) and (4.6) are only necessary conditions. Thus, for example, the crack surfaces are stress-free, that is, $\sigma_{r\varphi}(r, \pm\pi) = 0$, and equality (4.6) is satisfied for $\theta = \pm\pi$. However, $\sigma_{\varphi\varphi}(r, \pm\pi) = 0$ according to relation (3.2) and requirement (4.1) is clearly violated. On calculating the second derivative of function (4.3) in the case of constant d and transforming it, taking account of equilibrium equations (4.4), we have

$$F''=\frac{1}{d}J'=\frac{1}{d}\int_0^d\frac{\partial}{\partial\theta}\sigma_{sn}|_{n=0}ds-\frac{\partial}{\partial\theta}\sigma_{sn}|_{s=d,\,n=0}=\frac{1}{d}\int_0^d s\frac{\partial}{\partial n}\sigma_{sn}|_{n=0}ds-d\frac{\partial}{\partial n}\sigma_{sn}|_{s=d,\,n=0}=$$
$$=-\frac{1}{d}\int_0^d s\frac{\partial}{\partial s}\sigma_{ss}|_{n=0}ds-d\frac{\partial}{\partial s}\sigma_{ss}|_{s=d,\,n=0}=\frac{1}{d}\int_0^d\sigma_{ss}|_{n=0}ds-\left(\sigma_{ss}-d\frac{\partial}{\partial s}\sigma_{ss}\right)\Big|_{s=d,\,n=0} \tag{4.8}$$

At the point of a strictly local maximum, the quantity $F''(\tau_*;\theta)$, which only contains tensile stresses for directions perpendicular to the section $I(\theta)$, is found to be negative. The expression for the second derivative of function (4.2) in the variables $d(\theta)$ and $\Sigma_c(\theta)$ is extremely lengthy. Nevertheless, as in the case of a composite plane, the necessary and sufficient conditions for local maxima when $\pm\theta \in (0, \pi)$ can be employed, not forgetting to include $\theta = 0$ in the number of "suspicious" points.

5. CONSEQUENCES OF NOVOZHILOV'S CRITERION

If the dimension d is intrinsically small, then, with a certain error, the stresses $\sigma_{\varphi\varphi}(\tau_*; r, \varphi)$ and $\sigma_{r\varphi}(\tau_*; r, \varphi)$ in formulae (4.1) and (4.7) can be replaced [19] by the leading terms $r^{-1/2}\Sigma_{\varphi\varphi}(\tau_*; \varphi)$ and $r^{-1/2}\Sigma_{r\varphi}(\tau_*; \varphi)$ of their asymptotic expansion near the crack tip. As a result, the absolutely simple conditions

$$\frac{2}{\sqrt{d}}\Sigma_{\varphi\varphi}(\tau^*;\theta) = \sigma_c \tag{5.1}$$

$$\Sigma_{r\varphi}(\tau^*;\theta) = 0 \tag{5.2}$$

follow from equality (4.1) and, in the case of isotropic strength properties, also from equality (4.7). It is logical to refer to relation (5.1) as the *asymptotic form* of Novozhilov's criterion. After adding the condition for a global maximum at the point θ, it resembles the well known criterion for maximum tensile stresses and the necessary condition (5.2) is the criterion "$K_{\text{II}} = 0$" (a crack develops in the direction in which there are no shear forces). However, in the case of an open crack in an isotropic body, it is well known [30, 37, 38] that the above-mentioned criteria in the *a posteriori* formulation (the stress intensity factor K_{I} is the largest among those possible and the stress intensity factor K_{II} at the tip of the small crack branch is equal to zero) indicate a direction which is different from that obtained from formulae (5.1) and (5.2).

The sufficient condition for a local maximum $F''(\tau_*;\theta) < 0$ transforms, according to the calculation (4.8), into the inequality

$$\Sigma_{rr}(\tau_*;\theta) < 0$$

Hence, the direction in which a crack develops is characterized by the *maximum breaking* stress $r^{-1/2}\Sigma_{\varphi\varphi}(\tau_*;\theta)$, by *zero shear* stress $r^{-1/2}\Sigma_{r\varphi}(\tau_*;\theta)$ and by a *negative longitudinal* stress $r^{-1/2}\Sigma_{rr}(\tau_*;\theta)$.

We note that, in the paper by Morozov and Novozhilov [19], which analyses the direction of propagation of an open crack in an orthotropic material, it was not Novozhilov's criterion itself which was used but its asymptotic form (5.1). Here, the parameter d was assumed to be constant and the quantity $\sigma_c(\theta)$ was taken equal to $\sigma_{cx}\sin^2\theta + \sigma_{cy}\cos^2\theta$, that is, the reasons for the observed [19] deviation of the crack from a rectilinear path are solely due to the variability of the strength characteristic σ_c.

The need to involve polynomial asymptotic behaviour in the criterion for an incubation time [33], which extends criterion (4.1) to the case of dynamic fracture, is well known [34]. Since, according to equality (3.2), the existence of contact between the crack surfaces is not defined in many cases by the leading term in the asymptotic form of the stresses, taking account of the smaller terms becomes ever more important. We shall estimate the error in calculating the direction angle of the crack branch in homogeneous isotropic solid when the asymptotic criterion (5.1) is used.

The expansion of the stress field close to the crack tip O takes form

$$\sigma_{\varphi\varphi}(x) = -\frac{1}{4}\frac{K}{(2\pi r)^{1/2}}\left\{3\sin\frac{\varphi}{2} + 3\sin\frac{3\varphi}{2}\right\} + \sigma_{11}^0\sin^2\varphi + \sigma_{22}^0\cos^2\varphi + O(r^{1/2}) \tag{5.3}$$

$$\sigma_{r\varphi}(x) = \frac{1}{4}\frac{K}{(2\pi r)^{1/2}}\left\{\cos\frac{\varphi}{2} + 3\cos\frac{3\varphi}{2}\right\} + (\sigma_{22}^0 - \sigma_{11}^0)\sin\varphi\cos\varphi + O(r^{1/2}) \tag{5.4}$$

Here K is the stress intensity factor, which is assumed to be non-zero, σ_{11}^0 and σ_{22}^0 are finite components of the stresses at the crack tip, and $\sigma_{22}^0 \le 0$, in accordance with formulae (1.6) and (3.2). It is well known that Novozhilov's criterion indicates the following critical stress intensity factor and direction angle

$$|K^\star| = \frac{1}{4}\sigma_c(6\pi d)^{1/2}, \quad \theta^\star = -2\,\mathrm{sign}K\arcsin\frac{1}{\sqrt{3}} \tag{5.5}$$

Assuming that the magnitude of $\delta = |K|^{-1}|\sigma_{22}^0 - \sigma_{11}^0|d^{1/2}$ is small and using the necessary condition (4.7), we obtain the leading term of the correction to the direction angle

$$\theta^* = -\mathrm{sign}K\left\{2\arcsin\frac{1}{\sqrt{3}} + \frac{4}{27}(3\pi d)^{1/2}\frac{1}{K}(\sigma_{22}^0 - \sigma_{11}^0)\right\} \tag{5.6}$$

We now calculate the left-hand side of equality (4.1) in the asymptotic form (5.3) when $\theta = \theta^*$ and find that fracture does not occur when the following inequality is satisfied

$$-\frac{4}{\sqrt{3}}(2\pi d)^{-1/2}K + \frac{1}{9}(8\sigma_{11}^0 + \sigma_{22}^0) < \sigma_c \tag{5.7}$$

For unchanged K and $\sigma_{22}^0 \leq 0$ in the case of a tensile (compressive) stress along the crack, the direction angle decreases (increases). Naturally, formulae (5.6) and (5.7) are approximate, and their accuracy is determined by the quantities δ^2 and $\max_{r \leq d}\{|\tilde{\sigma}_{\varphi\varphi}(r, \varphi)| + |\tilde{\sigma}_{r\varphi}(r, \varphi)\}$, where $\tilde{\sigma}_{\varphi\varphi}$ and $\tilde{\sigma}_{r\varphi}$ are the residues in expansion (5.3) and (5.4).

We now return to asymptotic Novozhilov's criterion. One amusing fact follows from formula (5.2), which holds for a crack on the axis of elastic symmetry of a homogeneous medium. Since, by virtue of equality (3.2), the angular part $\Sigma_{\varphi\varphi}(\tau; \varphi)$ of the failure stresses vanishes when $\varphi = \pm\pi$, it reaches a maximum (or minimum) value at the point φ_* within the interval $(-\pi, \pi)$. In this case, $\Sigma_{\varphi r}(-\varphi; \varphi_*) = 0$, and, consequently, the angular part of the shear stress must change sign in the interval $(-\pi, \pi)$ for any anisotropic material. It has been mentioned that, according to conditions (5.1) and (5.2), for branching of a crack it is necessary that the angular part $\Sigma_{r\varphi}$ when $r^{-1/2}$ should vanish at least twice within the interval $(-\pi, \pi)$, changing its sign here from plus to minus. What has been said means that the function $\Sigma_{\varphi r}$ must have no less then five zeroes in the section $[-\pi, \pi]$. Naturally, a third degree harmonic polynomial of the variable $\varphi/2$ has no such property. So, branching of a crack does not occur within the framework of asymptotic Novozhilov's criterion, that is, in the case of an intrinsically small parameter d. For the same reason, a consideration of the binomial asymptotic form of the stresses (5.3) and (5.4) also does not help to reveal the possibility of the branching of a crack. This completely agrees with experimental fact: quasistatic branching of a crack (without the influence of dynamic effects) is not observed in a homogeneous isotropic brittle material.

The scheme of application of Novozhilov's criterion which has been described is also suitable in the case of an open crack. Naturally, it is necessary to change the asymptotic formulae (5.3) and (5.4) and to put $\sigma_{22}^0 = 0$ and introduce terms corresponding to the first (cleavage) mode and having positive stress intensity factors $K_1 > 0$.

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